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A Level Further Maths

Topic 1: Core Pure I: Proof, Complex Numbers and Matrices

AQA Further Maths (7367) | Mark Scheme | 100 marks

Instructions

- Answer **all** questions. You must show sufficient working - answers with no working may score no marks.
- You may use a calculator unless a question states otherwise.
- Give exact answers where required, otherwise to an appropriate degree of accuracy.
- There are 17 questions in this paper. The total mark is **100**.

1. This question is about the manipulation of complex numbers. Let $z_1 = 3 + 2i$ and $z_2 = 1 - 4i$.

(a) Find $z_1 + z_2$. **(1)**

✓ $(3 + 1) + (2 - 4)i = 4 - 2i$ (1)

(b) Find z_1z_2, giving your answer in the form $a + bi$. **(2)**

✓ $(3)(1) + (3)(-4i) + (2i)(1) + (2i)(-4i) = 3 - 12i + 2i - 8i^2$ (1)

✓ $= 3 - 10i + 8 = 11 - 10i$ (1)

(c) Find z_1/z_2, giving your answer in the form $a + bi$. **(3)**

✓ multiply numerator and denominator by conjugate $1 + 4i$: $(3 + 2i)(1 + 4i) / (1 - 4i)(1 + 4i)$ (1)

✓ numerator $= 3 + 12i + 2i + 8i^2 = -5 + 14i$; denominator $= 1 + 16 = 17$ (1)

✓ $z_1/z_2 = -5/17 + (14/17)i$ (1)

2. The complex number $w = -2 + 2i$ is plotted on an Argand diagram.

(a) Find the modulus $|w|$. **(2)**

✓ $|w| = \sqrt{((-2)^2 + 2^2)} = \sqrt{8}$ (1)

✓ $= 2\sqrt{2}$ (1)

(b) Find the argument $\arg(w)$, giving your answer in radians in the interval $(-\pi, \pi]$. **(2)**

✓ point lies in the second quadrant; reference angle $= \arctan(2/2) = \pi/4$ (1)

✓ $\arg(w) = \pi - \pi/4 = 3\pi/4$ (1)

(c) Hence write w in modulus-argument form. **(1)**

✓ $w = 2\sqrt{2}\,(\cos(3\pi/4) + i \sin(3\pi/4))$ (1)

3. Prove the following result by mathematical induction. For all positive integers n, the sum of the first n cubes satisfies the formula below.

(a) Prove by induction that $1^3 + 2^3 + 3^3 + ... + n^3 = (1/4)n^2(n + 1)^2$ for all integers n >= 1. **(6)**

✓ Base case n = 1: LHS = $1^3 = 1$; RHS = $(1/4)(1)^2(2)^2 = 1$, so true for n = 1 (1)
✓ Assume true for n = k: $1^3 + ... + k^3 = (1/4)k^2(k + 1)^2$ (1)
✓ Consider n = k + 1: sum = $(1/4)k^2(k + 1)^2 + (k + 1)^3$ (1)
✓ factor $(k + 1)^2$: $= (1/4)(k + 1)^2[k^2 + 4(k + 1)] = (1/4)(k + 1)^2(k^2 + 4k + 4)$ (1)
✓ $= (1/4)(k + 1)^2(k + 2)^2$, which is the formula with n = k + 1 (1)
✓ true for n = 1 and (true for k implies true for k + 1), so by induction true for all n >= 1 (1)

4. This question concerns the matrix A = (2 1 ; 1 3) (rows: top row 2, 1; bottom row 1, 3).

(a) Find the determinant of A. **(1)**

✓ det A = (2)(3) - (1)(1) = 6 - 1 = 5 (1)

(b) Find the inverse matrix A^{-1}. **(2)**

✓ A^{-1} = (1/det)(3 -1 ; -1 2) (1)
✓ A^{-1} = (1/5)(3 -1 ; -1 2), i.e. (3/5 -1/5 ; -1/5 2/5) (1)

(c) Hence solve the simultaneous equations 2x + y = 4 and x + 3y = -3. **(3)**

✓ write as A(x ; y) = (4 ; -3), so (x ; y) = A^{-1}(4 ; -3) (1)
✓ x = (1/5)(3(4) - 1(-3)) = (1/5)(12 + 3) = 15/5 = 3 (1)
✓ y = (1/5)(-1(4) + 2(-3)) = (1/5)(-4 - 6) = -10/5 = -2 ; so x = 3, y = -2 (1)

5. A linear transformation of the plane is represented by the 2x2 matrix M = (0 -1 ; 1 0) (top row 0, -1; bottom row 1, 0).

(a) Describe fully the geometric transformation represented by M. **(2)**

✓ rotation about the origin (1)
✓ through 90 degrees anticlockwise (1)

(b) Find the image of the point (3, 1) under this transformation. **(2)**

✓ M(3 ; 1) = (0(3) - 1(1) ; 1(3) + 0(1)) (1)
✓ = (-1 ; 3), i.e. the point (-1, 3) (1)

(c) Find the matrix that represents the transformation M applied twice (i.e. M^2), and describe it geometrically. **(3)**

✓ M^2 = (0 -1 ; 1 0)(0 -1 ; 1 0) = (-1 0 ; 0 -1) (1)
✓ = (-1 0 ; 0 -1) (1)
✓ this represents a rotation about the origin through 180 degrees (enlargement scale factor -1) (1)

6. The complex number z satisfies $z = 4(\cos(\pi/6) + i \sin(\pi/6))$. Use de Moivre's theorem where appropriate.

(a) Find z^3 in modulus-argument form. **(2)**

✓ by de Moivre: $z^3 = 4^3(\cos(3 \times \pi/6) + i \sin(3 \times \pi/6))$ (1)
✓ $= 64(\cos(\pi/2) + i \sin(\pi/2))$ (1)

(b) Hence express z^3 in the form a + bi. **(2)**

✓ $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$ (1)
✓ $z^3 = 64(0 + i) = 64i$ (1)

(c) Find $|z^5|$. **(1)**

✓ $|z^5| = 4^5 = 1024$ (1)

7. This question is about the cube roots of a complex number.

(a) Solve the equation $z^3 = 8$, giving the three roots in the form a + bi where appropriate. **(4)**

✓ write $8 = 8(\cos 0 + i \sin 0)$; roots have modulus $8^{1/3} = 2$ (1)
✓ arguments are $(0 + 2k\pi)/3$ for k = 0, 1, 2, i.e. 0, $2\pi/3$, $4\pi/3$ (1)
✓ root 1: $2(\cos 0 + i \sin 0) = 2$; root 2: $2(\cos(2\pi/3) + i \sin(2\pi/3)) = -1 + \sqrt{3}\, i$ (1)
✓ root 3: $2(\cos(4\pi/3) + i \sin(4\pi/3)) = -1 - \sqrt{3}\, i$ (1)

(b) State the geometric configuration of the three roots when plotted on an Argand diagram. **(2)**

✓ the three roots lie on a circle of radius 2 centred at the origin (1)
✓ they form the vertices of an equilateral triangle / are equally spaced at intervals of $2\pi/3$ (1)

8. The fifth roots of unity are the solutions of $z^5 = 1$.

(a) Write down the five fifth roots of unity in the form $e^{i\theta}$ (or $\cos\theta + i \sin\theta$). **(2)**

✓ roots are $z = \cos(2k\pi/5) + i \sin(2k\pi/5)$ for k = 0, 1, 2, 3, 4 (1)
✓ i.e. 1, $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{6\pi i/5}$, $e^{8\pi i/5}$ (1)

(b) Show that the sum of all five fifth roots of unity is zero. **(3)**

✓ the roots are a geometric series with first term 1 and common ratio $w = e^{2\pi i/5}$ (1)
✓ sum = $(w^5 - 1)/(w - 1)$, and $w^5 = 1$ so the numerator is 0 (1)
✓ since $w \neq 1$ the denominator is non-zero, hence the sum = 0 (1)

9. On an Argand diagram, points are represented by complex numbers z. Sketch and describe the loci defined below (no diagram required in your answer; give the geometric description and equation).

(a) Describe the locus $|z - 3i| = 2$. **(2)**

✓ a circle (1)
✓ centre (0, 3) i.e. the point 3i, radius 2 (1)

(b) Describe the locus $|z - 2| = |z + 4|$. **(3)**

✓ set of points equidistant from 2 (i.e. (2, 0)) and -4 (i.e. (-4, 0)) (1)
✓ this is the perpendicular bisector of the segment joining these two points (1)
✓ the bisector is the vertical line x = -1, i.e. Re(z) = -1 (1)

(c) Describe the locus $\arg(z - 1) = \pi/4$. **(2)**

✓ a half-line (ray) starting at the point (1, 0) (excluding that point) (1)
✓ making an angle of $\pi/4$ with the positive real direction (1)

10. The matrix B = (1 2 ; 2 4) (top row 1, 2; bottom row 2, 4) represents a transformation of the plane.

(a) Show that B is singular, and state what this implies about the transformation. **(2)**

✓ det B = (1)(4) - (2)(2) = 4 - 4 = 0, so B is singular (1)
✓ the transformation is not invertible; the plane is mapped onto a line through the origin (area scale factor 0) (1)

(b) Find the equation of the line onto which the whole plane is mapped by B. **(3)**

✓ image of (x ; y) is (x + 2y ; 2x + 4y) = (x + 2y)(1 ; 2) (1)
✓ so every image point is a scalar multiple of the vector (1, 2) (1)
✓ the image is the line y = 2x (1)

11. A transformation T of the plane is represented by the matrix C = (3 0 ; 0 3) multiplied on the left by the reflection R = (0 1 ; 1 0); that is, T = RC where C is the top row 3, 0 and bottom row 0, 3, and R has top row 0, 1 and bottom row 1, 0.

(a) Describe the transformation represented by C. **(1)**

✓ enlargement, centre the origin, scale factor 3 (1)

(b) Find the single matrix representing T = RC. **(2)**

✓ RC = (0 1 ; 1 0)(3 0 ; 0 3) (1)

✓ = (0 3 ; 3 0) (1)

(c) State a non-zero point that is invariant under the transformation T, or explain why none exists other than the origin. **(3)**

✓ invariant points satisfy T(x ; y) = (x ; y): (3y ; 3x) = (x ; y) (1)

✓ from 3y = x and 3x = y: substituting gives 9y = y, so y = 0 and hence x = 0 (1)

✓ therefore the only invariant point is the origin (0, 0) (1)

12. Use de Moivre's theorem to derive a multiple-angle identity.

(a) By expanding $(\cos\theta + i\sin\theta)^3$ and applying de Moivre's theorem, show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **(4)**

✓ by de Moivre, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ (1)

✓ expand binomially: $\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$ (1)

✓ real part = $\cos^3\theta - 3\cos\theta\sin^2\theta$, so $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ (1)

✓ use $\sin^2\theta = 1 - \cos^2\theta$: $\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) = 4\cos^3\theta - 3\cos\theta$ (1)

(b) Hence, using $3\theta = \pi/3$, show that $x = \cos(\pi/9)$ is a root of the cubic equation $8x^3 - 6x - 1 = 0$. **(2)**

✓ when $3\theta = \pi/3$, $\cos 3\theta = \cos(\pi/3) = 1/2$, so $4x^3 - 3x = 1/2$ (1)

✓ i.e. $8x^3 - 6x - 1 = 0$, with $x = \cos(\pi/9)$ a root (1)

13. Prove the following divisibility result by induction.

(a) Prove by induction that $5^n + 2(11^n)$ is divisible by 3 for all positive integers n. **(6)**

✓ Base case n = 1: 5 + 2(11) = 5 + 22 = 27 = 3 x 9, divisible by 3 (1)

✓ Assume true for n = k: $5^k + 2(11^k) = 3m$ for some integer m (1)

✓ Consider n = k + 1: $5^{k+1} + 2(11^{k+1}) = 5(5^k) + 22(11^k)$ (1)

✓ write as $5(5^k + 2(11^k)) + (22 - 10)(11^k) = 5(3m) + 12(11^k)$ (1)

✓ = $3(5m + 4(11^k))$, which is a multiple of 3 (1)

✓ true for n = 1, and (true for k implies true for k + 1), so by induction true for all n >= 1 (1)

14. The complex number $z = 1 + i\sqrt{3}$.

(a) Express z in modulus-argument form. **(3)**

✓ $|z| = \sqrt{(1^2 + (\sqrt{3})^2)} = \sqrt{(1 + 3)} = 2$ (1)

✓ arg z = $\arctan(\sqrt{3} / 1) = \pi/3$ (first quadrant) (1)

✓ $z = 2(\cos(\pi/3) + i\sin(\pi/3))$ (1)

(b) Use de Moivre's theorem to find z^6, giving your answer as a single real or imaginary number. **(3)**

✓ $z^6 = 2^6(\cos(6 \times \pi/3) + i\sin(6 \times \pi/3)) = 64(\cos 2\pi + i\sin 2\pi)$ (1)

✓ $\cos 2\pi = 1$, $\sin 2\pi = 0$ (1)

✓ $z^6 = 64$ (1)

15. This question concerns invariant lines of a transformation. The matrix D = (2 0 ; 0 3) (top row 2, 0; bottom row 0, 3) represents a stretch.

(a) Describe fully the transformation represented by D. **(2)**

✓ a stretch of scale factor 2 parallel to the x-axis (1)
✓ and a stretch of scale factor 3 parallel to the y-axis (1)

(b) Show that the x-axis (the line $y = 0$) is an invariant line under D. **(2)**

✓ a general point on $y = 0$ is $(x, 0)$; its image is D(x ; 0) = (2x ; 0) (1)
✓ the image $(2x, 0)$ still lies on $y = 0$, so the line is invariant (1)

(c) State the area scale factor of the transformation D, and justify your answer. **(2)**

✓ area scale factor = $|\det D| = |(2)(3) - 0| = 6$ (1)
✓ the determinant gives the factor by which areas are multiplied under the transformation (1)

16. The complex number $z = 2 - 2i$.

(a) Find the modulus and argument of z, giving the argument in the interval $(-\pi, \pi]$. **(3)**

✓ $|z| = \sqrt{(2^2 + (-2)^2)} = \sqrt{8} = 2\sqrt{2}$ (1)
✓ z lies in the fourth quadrant; reference angle = $\arctan(2/2) = \pi/4$ (1)
✓ $\arg z = -\pi/4$ (1)

(b) Hence, using de Moivre's theorem, find z^4 in the form $a + bi$. **(3)**

✓ $z^4 = (2\sqrt{2})^4(\cos(4 \times (-\pi/4)) + i \sin(4 \times (-\pi/4))) = 64(\cos(-\pi) + i \sin(-\pi))$ (1)
✓ $\cos(-\pi) = -1$, $\sin(-\pi) = 0$ (1)
✓ $z^4 = 64(-1 + 0i) = -64$ (1)

17. The complex number z satisfies the equation $z^4 = -16$.

(a) Express -16 in modulus-argument form, taking the principal argument. **(1)**

✓ $-16 = 16(\cos \pi + i \sin \pi)$ (1)

(b) Hence find all four solutions of $z^4 = -16$ in modulus-argument form. **(3)**

✓ modulus of each root = $16^{1/4} = 2$ (1)
✓ arguments = $(\pi + 2k\pi)/4$ for $k = 0, 1, 2, 3$, i.e. $\pi/4$, $3\pi/4$, $-3\pi/4$, $-\pi/4$ (1)
✓ roots: $2(\cos(\pi/4) + i \sin(\pi/4))$, $2(\cos(3\pi/4) + i \sin(3\pi/4))$, $2(\cos(-3\pi/4) + i \sin(-3\pi/4))$, $2(\cos(-\pi/4) + i \sin(-\pi/4))$ (1)

(c) Express the root with argument $\pi/4$ in the form $a + bi$. **(2)**

✓ $2(\cos(\pi/4) + i \sin(\pi/4)) = 2(\sqrt{2}/2 + i\sqrt{2}/2)$ (1)
✓ $= \sqrt{2} + \sqrt{2}\, i$ (1)