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A Level Further Maths

Topic 2: Core Pure II: Calculus, Vectors, Polar and Differential Equations

AQA Further Maths (7367)	Mark Scheme	100 marks

Instructions

- Answer **all** questions. You must show sufficient working - answers with no working may score no marks.
- You may use a calculator unless a question states otherwise.
- Give exact answers where required, otherwise to an appropriate degree of accuracy.
- There are 17 questions in this paper. The total mark is **100**.

1. This question is about hyperbolic functions. Recall that cosh x = $(e^x + e^{-x})/2$ and sinh x = $(e^x - e^{-x})/2$.

(a) Find the exact value of cosh(0). **(1)**

✓ cosh 0 = $(e^0 + e^0)/2 = (1 + 1)/2 = 1$ (1)

(b) Prove the identity $\cosh^2 x - \sinh^2 x = 1$. **(3)**

✓ $\cosh^2 x = (e^x + e^{-x})^2/4 = (e^{2x} + 2 + e^{-2x})/4$ (1)
✓ $\sinh^2 x = (e^x - e^{-x})^2/4 = (e^{2x} - 2 + e^{-2x})/4$ (1)
✓ $\cosh^2 x - \sinh^2 x = (e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x})/4 = 4/4 = 1$ (1)

2. This question concerns differentiation of hyperbolic functions.

(a) Show, from the exponential definition, that d/dx (sinh x) = cosh x. **(2)**

✓ $d/dx[(e^x - e^{-x})/2] = (e^x + e^{-x})/2$ (1)
✓ = cosh x (1)

(b) Hence find d/dx (sinh(3x)). **(2)**

✓ by the chain rule, d/dx(sinh 3x) = cosh(3x) x 3 (1)
✓ = 3 cosh 3x (1)

3. This question is about polar coordinates. A curve has polar equation r = 2 + 2cos θ for 0 <= θ < 2π (a cardioid).

(a) Find the value of r when θ = 0 and when θ = π. **(2)**

✓ at θ = 0: r = 2 + 2cos 0 = 2 + 2 = 4 (1)
✓ at θ = π: r = 2 + 2cos π = 2 - 2 = 0 (1)

(b) State the Cartesian coordinates of the point on the curve where θ = π/2. **(2)**

✓ at θ = π/2: r = 2 + 2cos(π/2) = 2 (1)
✓ x = r cos θ = 0, y = r sin θ = 2, so the point is (0, 2) (1)

4. This question concerns improper integrals.

(a) Evaluate the improper integral ∫ from 1 to ∞ of $(1/x^2)$ dx, or state that it diverges. **(3)**

✓ $\int x^{-2}$ dx = -1/x ; evaluate as a limit: lim (t→∞) [-1/x] from 1 to t (1)
✓ = lim (t→∞) (-1/t + 1) (1)
✓ = 0 + 1 = 1 (converges) (1)

(b) Determine whether ∫ from 1 to ∞ of (1/x) dx converges or diverges, justifying your answer. **(2)**

✓ ∫ (1/x) dx = ln x ; lim (t→∞) [ln x] from 1 to t = lim (t→∞)(ln t - 0) (1)
✓ ln t → ∞ as t → ∞, so the integral diverges (1)

5. The region R is bounded by the curve $y = x^2$, the x-axis, and the line x = 2. The region is rotated 2π radians about the x-axis.

(a) Write down an integral expression for the volume of the solid of revolution generated. **(2)**

✓ V = π ∫ from 0 to 2 of y^2 dx = π ∫ from 0 to 2 of $(x^2)^2$ dx (1)
✓ = π ∫ from 0 to 2 of x^4 dx (1)

(b) Hence evaluate the volume, giving your answer as an exact multiple of π. **(3)**

✓ π $[x^5/5]$ from 0 to 2 (1)
✓ = π $(2^5/5 - 0)$ = π(32/5) (1)
✓ V = 32π/5 (1)

6. This question is about the method of differences.

(a) Show that 1/(r(r + 1)) = 1/r - 1/(r + 1). **(2)**

✓ 1/r - 1/(r + 1) = ((r + 1) - r)/(r(r + 1)) (1)
✓ = 1/(r(r + 1)) as required (1)

(b) Hence use the method of differences to find the sum from r = 1 to n of 1/(r(r + 1)). **(3)**

✓ sum = Σ (1/r - 1/(r + 1)) telescopes: (1 - 1/2) + (1/2 - 1/3) + ... + (1/n - 1/(n + 1)) (1)
✓ all interior terms cancel, leaving 1 - 1/(n + 1) (1)
✓ = n/(n + 1) (1)

(c) Deduce the sum to infinity of the series in part (b). **(1)**

✓ as n → ∞, n/(n + 1) → 1, so the sum to infinity is 1 (1)

7. This question is about lines and planes in three dimensions. The line L has vector equation r = (1, 0, 2) + λ(2, 1, -1), and the plane Π has equation 2x + y - z = 5.

(a) Write down a normal vector to the plane Π. **(1)**

✓ normal vector n = (2, 1, -1) (1)

(b) Find the coordinates of the point where the line L meets the plane Π. **(4)**

✓ point on L: (1 + 2λ, λ, 2 - λ); substitute into plane equation (1)
✓ 2(1 + 2λ) + λ - (2 - λ) = 5 (1)
✓ 2 + 4λ + λ - 2 + λ = 5, so 6λ = 5, λ = 5/6 (1)
✓ point = (1 + 5/3, 5/6, 2 - 5/6) = (8/3, 5/6, 7/6) (1)

(c) Find the acute angle between the line L and the plane Π. Give your answer in degrees to 1 decimal place. **(4)**

✓ direction of L is d = (2, 1, -1); use sin α = |d.n| / (|d||n|) (1)
✓ d.n = (2)(2) + (1)(1) + (-1)(-1) = 4 + 1 + 1 = 6 (1)
✓ |d| = √6, |n| = √6, so sin α = 6/(√6 x √6) = 6/6 = 1 (1)
✓ α = 90 degrees (the line is perpendicular to the plane, i.e. parallel to the normal) (1)

8. This question is about further vectors. Points A, B and C have position vectors a = (1, 2, 0), b = (3, 1, 2) and c = (2, 4, 1).

(a) Find the vector AB. **(2)**

✓ AB = b - a = (3 - 1, 1 - 2, 2 - 0) (1)
✓ = (2, -1, 2) (1)

(b) Find AB . AC (the scalar product of AB with AC), where AC = c - a. **(3)**

✓ AC = c - a = (2 - 1, 4 - 2, 1 - 0) = (1, 2, 1) (1)
✓ AB . AC = (2)(1) + (-1)(2) + (2)(1) (1)
✓ = 2 - 2 + 2 = 2 (1)

(c) Hence find the angle BAC, giving your answer in degrees to 1 decimal place. **(3)**

✓ |AB| = √(4 + 1 + 4) = 3, |AC| = √(1 + 4 + 1) = √6 (1)
✓ cos(BAC) = (AB.AC)/(|AB||AC|) = 2/(3√6) (1)
✓ BAC = arccos(2/(3√6)) = arccos(0.2722) = 74.2 degrees (1)

9. This question is about first-order differential equations. Solve the equation $dy/dx = 3x^2y$ given the boundary condition y = 2 when x = 0.

(a) Separate the variables and integrate to find the general solution. **(4)**

✓ $(1/y)\ dy = 3x^2\ dx$ (1)
✓ $\int (1/y)\ dy = \int 3x^2\ dx$ (1)
✓ $\ln|y| = x^3 + c$ (1)
✓ $y = A\ e^{x^3}$ where $A = e^c$ (1)

(b) Use the boundary condition to find the particular solution. **(2)**

✓ at x = 0, y = 2: $2 = A\ e^0 = A$, so A = 2 (1)
✓ $y = 2\ e^{x^3}$ (1)

10. This question is about second-order differential equations with constant coefficients. Consider the equation $d^2y/dx^2 - 5\ dy/dx + 6y = 0$.

(a) Write down the auxiliary (characteristic) equation and solve it. **(3)**

✓ auxiliary equation: $m^2 - 5m + 6 = 0$ (1)
✓ factorise: (m - 2)(m - 3) = 0 (1)
✓ m = 2 or m = 3 (1)

(b) Hence write down the general solution of the differential equation. **(2)**

✓ roots are real and distinct, so general solution is $y = Ae^{2x} + Be^{3x}$ (1)
✓ where A and B are arbitrary constants (1)

11. A particle moves in a straight line and its displacement x metres from a fixed point O at time t seconds satisfies the equation $d^2x/dt^2 = -9x$ (simple harmonic motion).

(a) Write down the angular frequency ω and the period of the motion. **(2)**

✓ comparing with $d^2x/dt^2 = -\omega^2 x$ gives $\omega^2 = 9$, so ω = 3 (1)

✓ period T = 2π/ω = 2π/3 s (1)

(b) The general solution is x = A cos(3t) + B sin(3t). Initially (t = 0) the particle is at x = 4 with velocity dx/dt = 0. Find A and B. **(3)**

✓ at t = 0, x = 4: A cos 0 + B sin 0 = A = 4 (1)

✓ dx/dt = -3A sin(3t) + 3B cos(3t); at t = 0 this equals 3B = 0, so B = 0 (1)

✓ therefore A = 4, B = 0 and x = 4 cos(3t) (1)

(c) State the amplitude of the motion and the maximum speed of the particle. **(2)**

✓ amplitude = 4 m (1)

✓ maximum speed = ω x amplitude = 3 x 4 = 12 m s^{-1} (1)

12. This question is about damped oscillations. The displacement of a damped system satisfies $d^2x/dt^2 + 4\, dx/dt + 13x = 0$.

(a) Solve the auxiliary equation and state the type of damping. **(3)**

✓ auxiliary equation: $m^2 + 4m + 13 = 0$; m = (-4 ± √(16 - 52))/2 (1)

✓ = (-4 ± √(-36))/2 = -2 ± 3i (complex roots) (1)

✓ complex roots imply (light/under-) damping, i.e. oscillatory decay (1)

(b) Write down the general solution of the differential equation. **(2)**

✓ with roots -2 ± 3i, general solution is $x = e^{-2t}(A \cos 3t + B \sin 3t)$ (1)

✓ where A and B are arbitrary constants (1)

(c) State the long-term behaviour of x as t → ∞, with justification. **(1)**

✓ the factor $e^{-2t} \to 0$, so x → 0; the oscillations decay to zero (1)

13. This question is about the Maclaurin series.

(a) Write down the first three non-zero terms of the Maclaurin series for e^x. **(1)**

✓ $e^x = 1 + x + x^2/2! + \dots$ (1)

(b) Derive the Maclaurin series for sin x up to and including the term in x^5, by evaluating successive derivatives at x = 0. **(4)**

✓ f(x) = sin x, f(0) = 0; f'(x) = cos x, f'(0) = 1 (1)

✓ f''(x) = -sin x, f''(0) = 0; f'''(x) = -cos x, f'''(0) = -1 (1)

✓ $f^{(4)}(0) = 0$, $f^{(5)}(0) = 1$ (1)

✓ $\sin x = x - x^3/3! + x^5/5! = x - x^3/6 + x^5/120$ (1)

(c) Hence write down an approximation for sin(0.1) using terms up to x^3, giving your answer to 5 decimal places. **(2)**

✓ $\sin(0.1) \approx 0.1 - (0.1)^3/6 = 0.1 - 0.001/6$ (1)

✓ = 0.1 - 0.0001667 = 0.09983 (1)

14. This question concerns arc length. A curve has equation $y = (2/3)x^{3/2}$ for $0 <= x <= 3$.

(a) Show that $dy/dx = x^{1/2}$ and hence that $1 + (dy/dx)^2 = 1 + x$. **(2)**

✓ $dy/dx = (2/3)(3/2)x^{1/2} = x^{1/2}$ (1)
✓ so $(dy/dx)^2 = x$ and $1 + (dy/dx)^2 = 1 + x$ (1)

(b) Find the arc length of the curve from $x = 0$ to $x = 3$ using $s = \int \sqrt{(1 + (dy/dx)^2)}\, dx$. **(4)**

✓ $s = \int$ from 0 to 3 of $\sqrt{(1 + x)}\, dx$ (1)
✓ $= [(2/3)(1 + x)^{3/2}]$ from 0 to 3 (1)
✓ $= (2/3)(4^{3/2}) - (2/3)(1^{3/2}) = (2/3)(8) - (2/3)(1)$ (1)
✓ $= 16/3 - 2/3 = 14/3$ (1)

15. A second-order differential equation has a non-zero right-hand side: $d^2y/dx^2 - 3\, dy/dx + 2y = 4$.

(a) Find the complementary function. **(3)**

✓ auxiliary equation $m^2 - 3m + 2 = 0$ (1)
✓ $(m - 1)(m - 2) = 0$, so $m = 1$ or $m = 2$ (1)
✓ complementary function $y_c = Ae^x + Be^{2x}$ (1)

(b) Find a particular integral, and hence write the general solution. **(3)**

✓ try constant $y = k$; then $y'' = y' = 0$, so $2k = 4$, $k = 2$ (1)
✓ particular integral $y_p = 2$ (1)
✓ general solution $y = Ae^x + Be^{2x} + 2$ (1)

16. This question concerns the area enclosed by a polar curve. A curve has polar equation $r = 3 \sin \theta$ for $0 <= \theta <= \pi$.

(a) Write down the formula for the area enclosed by a polar curve between $\theta = \alpha$ and $\theta = \beta$, and write the integral needed for this curve over $0 <= \theta <= \pi$. **(2)**

✓ area $= (1/2) \int$ from α to β of $r^2\, d\theta$ (1)
✓ here area $= (1/2) \int$ from 0 to π of $(3 \sin \theta)^2\, d\theta = (1/2) \int$ from 0 to π of $9 \sin^2\theta\, d\theta$ (1)

(b) Using the identity $\sin^2\theta = (1 - \cos 2\theta)/2$, evaluate this area exactly. **(4)**

✓ area $= (9/2) \int$ from 0 to π of $(1 - \cos 2\theta)/2\, d\theta = (9/4) \int$ from 0 to π of $(1 - \cos 2\theta)\, d\theta$ (1)
✓ $= (9/4)[\theta - (1/2)\sin 2\theta]$ from 0 to π (1)
✓ $= (9/4)[(\pi - 0) - (0 - 0)]$ (1)
✓ $= 9\pi/4$ (1)

17. This question is about the volume of revolution about the y-axis. The region bounded by $y = x^2$, the y-axis and the line $y = 4$ is rotated 2π radians about the y-axis.

(a) Write down the integral for the volume using $V = \pi \int x^2\, dy$, expressing x^2 in terms of y. **(2)**

✓ since $y = x^2$, we have $x^2 = y$ (1)
✓ $V = \pi \int$ from 0 to 4 of $y\, dy$ (1)

(b) Hence evaluate the volume exactly. **(2)**

✓ $V = \pi\, [y^2/2]$ from 0 to 4 $= \pi(16/2 - 0)$ (1)
✓ $= 8\pi$ (1)

(c) The cylinder formed by rotating the rectangle $0 <= x <= 2$, $0 <= y <= 4$ about the y-axis has volume 16π. State the volume of the region between this cylinder and the solid found in part (b). **(2)**

✓ required volume = cylinder volume - solid volume $= 16\pi - 8\pi$ (1)
✓ $= 8\pi$ (1)